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Greedy F-colorings of graphs

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Dedicated to Frank Harary on the occasion of his 80th birthday

Abstract

Let $G=(V,E)$ be a connected graph. For a symmetric, integer-valued function δ on $V\times V$, where K is an integer constant, N_0 is the set of nonnegative integers, and $\boldsymbol{Z}$ is the set of integers, we define a C-mapping $F: V\times V\times N_0 \to \boldsymbol{Z}$ by $F(u,v,m)=\delta(u,v)+m-K$. A coloring c of G is an F-coloring if $F(u,v,|c(u)-c(v)|)\geqslant 0$ for every two distinct vertices u and v of G. The maximum color assigned by c to a vertex of G is the value of c, and the F-chromatic number $F(G)$ is the minimum value among all F-colorings of G. For an ordering s: $v_1, v_2, \ldots, v_n$ of the vertices of G, a greedy F-coloring c of s is defined by (1) $c(v_1)=1$ and (2) for each i with $1\leqslant i<n$, $c(v_{i+1})$ is the smallest positive integer p such that $F(v_j, v_{i+1}, |c(v_j)-p|)\geqslant 0$, for each j with $1\leqslant j\leqslant i$. The greedy F-chromatic number $gF(s)$ of s is the maximum color assigned by c to a vertex of G. The greedy F-chromatic number of G is $gF(G)=\min\{gF(s)\}$ over all orderings s of V. The Grundy F-chromatic number is $GF(G)=\max\{gF(s)\}$ over all orderings s of V. It is shown that $gF(G)=F(G)$ for every graph G and every F-coloring defined on G. The parameters $gF(G)$ and $GF(G)$ are studied and compared for a special case of the C-mapping F on a connected graph G, where $\delta(u,v)$ is the distance between u and v and $K=1+\operatorname{diam} G$.
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1. Introduction

Let G be a connected graph of order n and diameter d. For an integer k with $1 \leqslant k \leqslant d$, a *radio k-coloring* of G was defined in [1] as an assignment c of colors (positive integers) to the vertices of G such that

$$d(u,v) + |c(u) - c(v)| \geqslant 1 + k$$

for every two distinct vertices u and v of G. The *value* $\mathrm{rc}_k(c)$ of a radio k-coloring c of G is the maximum color assigned to a vertex of G, and the *radio k-chromatic number* $\mathrm{rc}_k(G)$ of G is $\min\{\mathrm{rc}_k(c)\}$ over all radio k-colorings c of G. A radio k-coloring c of G is a *minimum radio k-coloring* if $\mathrm{rc}_k(c) = \mathrm{rc}_k(G)$. Since $\mathrm{rc}_1(G)$ is the chromatic number $\chi(G)$, radio k-colorings provide a generalization of ordinary colorings of graphs. The *radio d-chromatic number* of a graph G has also been called the *radio number* and written as $\mathrm{rn}(G)$. Since no two vertices can be colored the same in a radio d-coloring, such a coloring is also a labeling, referred to as a *radio labeling*. Thus, in a radio labeling of a connected graph of diameter d, the labels assigned to two vertices whose distance is i must differ by at least $d-i+1$ for all i with $1 \leqslant i \leqslant d$. Radio 2-colorings have been called simply radio colorings in [10] and $L(2,1)$-colorings in [7–9].

For distinct vertices u and v in a connected graph G of order n, let $D(u,v)$ denote the length of a longest $u-v$ path in G. (Note that obtaining a longest path is, in general, NP-hard.) A coloring c of G is called a *hamiltonian coloring* in [3] if

$$D(u,v) + |c(u) - c(v)| \geqslant n - 1$$

for every two distinct vertices u and v of G. Thus, in a hamiltonian coloring of G, two vertices u and v can be assigned the same color only if G contains a hamiltonian $u-v$ path. The *value* $\mathrm{hc}(c)$ of a hamiltonian coloring c of G is the maximum color assigned to a vertex of G. The *hamiltonian chromatic number* $\mathrm{hc}(G)$ of G is $\min\{\mathrm{hc}(c)\}$ over all hamiltonian colorings c of G. In [3], hamiltonian chromatic numbers of some special classes of graphs were determined and a sharp upper bound for the hamiltonian chromatic number of a connected graph in terms of its order was established. Also, it was shown that for every two integers k and n with $k \geqslant 1$ and $n \geqslant 3$, there exists a hamiltonian graph of order n with hamiltonian chromatic number k if and only if $1 \leqslant k \leqslant n-2$.

While radio k-colorings of a connected graph were studied in [1,2], hamiltonian colorings were investigated in [3,4]. Radio k-colorings, including radio labelings (where the vertices are then assigned distinct positive integers), and hamiltonian colorings belong to a general class of colorings of connected graphs which we now describe.

Let $G = (V,E)$ be a connected graph and let c be a coloring (an assignment of positive integers) of G. Furthermore, let δ be a symmetric, integer-valued function on $V \times V$, that is, $\delta(u,v) = \delta(v,u)$ for all $u,v \in V$. For such a function δ and an integer constant K, we define a function $F : V \times V \times N_0 \to \mathbf{Z}$ by

$$F(u,v,m) = \delta(u,v) + m - K,$$

where N_0 is the set of nonnegative integers and $\mathbf{Z}$ is the set of integers. We refer to such a function as a *C-mapping*. If the coloring c satisfies

$$F(u,v,|c(u)-c(v)|) \geqslant 0$$

for every two distinct vertices u and v of G, then c is called an *F-coloring of G*. For example, if $\delta(u,v)=d(u,v)$, the standard distance on V, and $K=k+1$, where $1 \leqslant k \leqslant \operatorname{diam} G$, then c is a radio k-coloring; while if $\delta(u,v)=d(u,v)$ and $K=\operatorname{diam} G+1$, then c is a radio labeling. Indeed, if $\delta(u,v)=d(u,v)$ and $K=2$, then c is an ordinary coloring of G. Moreover, if $\delta(u,v)=D(u,v)$ and $K=n-1$, where n is the order of G, then c is a hamiltonian coloring of G. For a connected graph G and a C-mapping F on G, the *value* $F(c)$ of an F-coloring c of G is the maximum color assigned by c to a vertex of G. The *F-chromatic number* $F(G)$ of G is $\min\{F(c)\}$ over all F-colorings c of G. An F-coloring c of G is a *minimum F-coloring* if $F(c)=F(G)$.

2. The greedy F-chromatic number of a graph

It is well known that determining the ordinary chromatic number of a graph is, in general, extraordinarily difficult; in fact, it is NP-hard. An ordinary coloring of G can be produced in a greedy manner, that is, let $v_1,v_2,\ldots,v_n$ be an ordering of the vertices of G and assign v_1 the color 1. Once colors have been assigned to $v_1,v_2,\ldots,v_i$, where $1 \leqslant i < n$, we assign the smallest color to v_{i+1} that has not been used for any vertex adjacent to v_{i+1}. In this manner, we obtain an ordinary coloring of G. Whether the largest color assigned to a vertex of G is $\chi(G)$ or exceeds $\chi(G)$ depends on both G and the given ordering of the vertices. It is well known, however, that for every graph G, there exists an ordering of vertices of G such that the largest color assigned to a vertex of G by the greedy coloring algorithm is $\chi(G)$. The maximum color assigned to a vertex of G by the greedy coloring algorithm over all orderings of $V(G)$ has been called the Grundy number $\Gamma(G)$ and has been studied (see [5,6,11] for example). In the same manner, we can produce greedy F-colorings of G. For a connected graph G of order n, let $s: v_1,v_2,\ldots,v_n$ be an ordering of the vertices of G. For a C-mapping F on G, we define a coloring c of the vertices of G by

(1) $c(v_1)=1$,
(2) for each i with $1 \leqslant i < n$, $c(v_{i+1})$ is the smallest positive integer p such that $F(v_j,v_{i+1},|c(v_j)-p|) \geqslant 0$ for each j with $1 \leqslant j \leqslant i$.

Clearly, c is an F-coloring of G. An F-coloring obtained in this manner is called a *greedy F-coloring* of G. The *greedy F-chromatic number* $gF(s)$ of s is the maximum color assigned by c to a vertex of G. The greedy F-chromatic number $gF(G)$ of G is $\min\{gF(s)\}$, over all orderings s of V. The *Grundy F-chromatic number* $GF(G)$ of G is defined as $\max\{gF(s)\}$, over all orderings s of V.

We now show that the greedy F-chromatic number of G equals the F-chromatic number of G for every graph G and every C-mapping F defined on G. It is convenient to introduce some additional notation. Let $G=(V,E)$ be a graph of order n and let

N be the set of positive integers. For a function $f:V \to N$ and an integer i with $1 \leqslant i \leqslant n$, let

$$\sigma_i(f) = \min\left\{\sum_{u\in U} f(u)\colon\ U \subseteq V \text{ and } |U| = i\right\}.$$

For a nonempty set S of functions from V to N and an integer i with $1 \leqslant i \leqslant n$, we let

$$\sigma_i(S) = \min\{\sigma_i(f)\colon\ f \in S\}.$$

Theorem 2.1. *For every connected graph G and every C-mapping F on G,*

$$gF(G) = F(G).$$

Proof. Consider a connected graph G and an arbitrary C-mapping F on G. It suffices to show that there exists an ordering s of the vertices of G such that $gF(s) = F(G)$. Let $\mathscr{R}$ be the set of minimum F-colorings of G. Define a sequence $\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_n$ of subsets of $\mathscr{R}$ as follows:

(1) $\mathscr{R}_1$ is the subset of $\mathscr{R}$ consisting of those minimum F-colorings c in $\mathscr{R}$ for which there is some vertex u_1 in G such that $c(u_1) = 1$;
(2) for each i with $2 \leqslant i \leqslant n$, $\mathscr{R}_i$ is the subset of $\mathscr{R}_{i-1}$ consisting of those minimum F-colorings c in $\mathscr{R}_{i-1}$ such that $\sigma_i(c) = \sigma_i(R_{i-1})$.

Hence $\mathscr{R}_n \subseteq \mathscr{R}_{n-1} \subseteq \mathscr{R}_{n-2} \subseteq \cdots \subseteq \mathscr{R}_1 = \mathscr{R}$. Let $c^* \in \mathscr{R}_n$ and let $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $c^*(v_1) \leqslant c^*(v_2) \leqslant \cdots \leqslant c^*(v_n)$. Consider the ordering s: $v_1, v_2, \ldots, v_n$ of $V(G)$. We show that c^* is the greedy F-coloring of s. Assume, to the contrary, that c^* is not the greedy F-coloring of s. Let c' be the greedy F-coloring of s and so $c^* \neq c'$. Certainly, $c^*(v_1) = c'(v_1) = 1$. Let t be the smallest positive integer such that $c^*(v_t) \neq c'(v_t)$. Then $t \geqslant 2$. Since c' is the greedy F-coloring of s, it follows that $c'(v_t) < c^*(v_t)$. Define a coloring c'' of the vertices of G by $c''(v_t) = c'(v_t)$ and $c''(v_i) = c^*(v_i)$ for all $i \neq t$ and $1 \leqslant i \leqslant n$. We show that c'' is in fact an F-coloring of G. It suffices to show that $F(v_i, v_t, |c''(v_i) - c''(v_t)|) \geqslant 0$ for each $i \neq t$ and $1 \leqslant i \leqslant n$. We consider two cases.

Case 1: $i < t$. Since c' is the greedy F-coloring of s and $c^*(v_i) = c'(v_i)$ for all i with $1 \leqslant i \leqslant t-1$, it follows that

$$\begin{aligned} F(v_i, v_t, |c''(v_i) - c''(v_t)|) &= F(v_i, v_t, |c'(v_t) - c^*(v_i)|) \\ &= F(v_i, v_t, |c'(v_t) - c'(v_i)|) \geqslant 0. \end{aligned}$$

Case 2: $i > t$. Since $c^*(v_i) \geqslant c^*(v_t) > c'(v_t)$ and c^* is an F-coloring of G,

$$\begin{aligned} F(v_i, v_t, |c''(v_t) - c''(v_i)|) &= F(v_i, v_t, |c'(v_t) - c^*(v_i)|) \\ &= F(v_i, v_t, (c^*(v_i) - c'(v_t))) \\ &> F(v_i, v_t, (c^*(v_i) - c^*(v_t))) \geqslant 0. \end{aligned}$$

Thus c'' is an F-coloring of G. Since c^* is a minimum F-coloring of G and $c''(v_t) < c^*(v_t)$, we have $t < n$. It follows that c'' is a minimum F-coloring of G and so c'' is in R. By virtue of the definitions of R_{t-1} and of c'', we have $c'' \in R_{t-1}$. Since $\sigma_t(c'') < \sigma_t(c^*) = \sigma_t(R_{t-1})$, a contradiction is produced. □

The proof of Theorem 2.1 depends only on the fact that the function $F : V \times V \times N_0 \to \mathbf{Z}$ satisfies the two properties

(a) $F(u, v, m) = F(v, u, m)$ for all $u, v \in V$ and $m \in N_0$,
(b) if $m > m'$, then $F(u, v, m) > F(v, u, m')$ for all $u, v \in V$ and $m, m' \in N_0$.

For any such function F, we can define a coloring c of G as an *F-coloring* if c satisfies $F(u, v, |c(u) - c(v)|) \geqslant 0$ for every two distinct vertices u and v of G. By defining the F-chromatic number $F(G)$ and the greedy F-chromatic number $\mathrm{gF}(G)$ of G in the expected manner, we can obtain the following general result by the same argument used to prove Theorem 2.1.

Theorem 2.2. *Let $G = (V, E)$ be a connected graph and let $F : V \times V \times N_0 \to \mathbf{Z}$ be a function satisfying properties (a) and (b). Then $\mathrm{gF}(G) = F(G)$ for every F-coloring on G.*

3. The greedy radio number of a graph

In this section, we consider greedy F-colorings of connected graphs for the special case of the C-mapping F for which $\delta(u, v) = d(u, v)$ and $K = \operatorname{diam} G + 1$, that is, we consider greedy radio labelings. To review the situation in this special case, for a connected graph G of order n and diameter d, let s: $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G. We define a radio labeling c of G by

(1) $c(v_1) = 1$,
(2) for $1 \leqslant i < n$, $c(v_{i+1})$ is the smallest positive integer p such that $d(u, v_{i+1}) + |c(u) - p| \geqslant 1 + d$ for all $u \in \{v_1, v_2, \ldots, v_i\}$.

The radio labeling c is then the *greedy radio labeling* of s and the *greedy radio number* $\mathrm{gn}(s)$ of s is the maximum color assigned by c to a vertex of G. The greedy radio number $\mathrm{gn}(G)$ of G is $\min\{\mathrm{gn}(s)\}$ over all orderings s of $V(G)$ and the *Grundy radio number* $\mathrm{GN}(G)$ of G is defined as $\max\{\mathrm{gn}(s)\}$ over all orderings s of $V(G)$. By Theorem 2.1, $\mathrm{gn}(G) = \mathrm{rn}(G)$ for every connected graph G.

For example, consider the path P_3 of order 3, which is known to have radio number 4. Fig. 1 shows all three orderings s_1, s_2, and s_3 of $V(P_3)$ with the corresponding greedy radio labelings. Then $\mathrm{gn}(s_1) = 5$ and $\mathrm{gn}(s_2) = \mathrm{gn}(s_3) = 4$. Thus $\mathrm{gn}(P_3) = \mathrm{rn}(P_3) = 4$ and $\mathrm{GN}(P_3) = 5$.

Of course, $\mathrm{rn}(G) \geqslant n$ for every connected graph G of order n. Graphs G of order n and diameter 2 for which $\mathrm{rn}(G) = n$ were characterized in [1]. A graph is *traceable* if it contains a hamiltonian path.

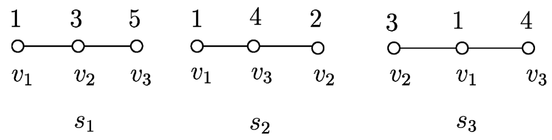

Fig. 1. The orderings s_1, s_2, and s_3 of $V(P_3)$.

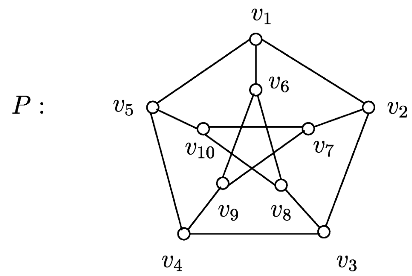

Fig. 2. The Petersen graph.

Theorem A. *Let G be a connected graph of order n and diameter* 2. *Then* $\mathrm{rn}(G) = n$ *if and only if $\bar{G}$ is traceable.*

The Petersen graph P shown in Fig. 2 has order 10, diameter 2, and radio number 10. Consider the three orderings of the vertices of P

s_1: $v_1, v_7, v_5, v_9, v_{10}, v_6, v_2, v_8, v_3, v_4$,

s_2: $v_1, v_5, v_4, v_9, v_2, v_{10}, v_7, v_8, v_6, v_3$,

s_3: $v_1, v_7, v_5, v_9, v_{10}, v_6, v_2, v_8, v_3, v_4$.

It can be verified that $\mathrm{gn}(s_1) = 10, \mathrm{gn}(s_2) = 11$, and $\mathrm{gn}(s_3) = 12$.

Since the Petersen graph P has order 10 and $\mathrm{gn}(s_1) = 10$, it follows that $\mathrm{gn}(P) \leqslant 10$. Also, P has diameter 2 and $\bar{P}$ is traceable. Thus, by Theorem A, $\mathrm{rn}(P) = 10$. Since $\mathrm{gn}(P) = \mathrm{rn}(P)$, it follows that $\mathrm{gn}(P) = 10$. Because there are orderings of $V(P)$, namely s_2 and s_3, having greedy radio number exceeding 10, it follows that $\mathrm{GN}(P) > 10$. In fact, $\mathrm{GN}(P) \geqslant 12$. Whether $\mathrm{GN}(P) = 12$ is not known. Observe that the complement $\bar{P}$ of the Petersen graph (which has diameter 2) contains the path $v_3, v_7, v_8, v_9, v_{10}, v_4, v_2$, which cannot be extended to a hamiltonian path in $\bar{P}$. This observation alone guarantees that $\mathrm{GN}(P) > 10$.

Theorem 3.1. *Let G be a connected graph of order n, diameter* 2, *and radio number n. If $\bar{G}$ contains a path that cannot be extended to a hamiltonian path in $\bar{G}$, then* $\mathrm{GN}(G) > \mathrm{gn}(G)$.

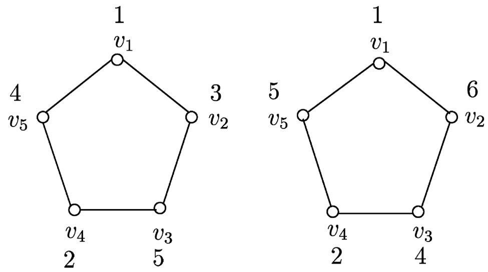

Fig. 3. The greedy labelings c and c'.

Proof. It suffices to show that there exist two orderings s and s' of vertices of G such that $\mathrm{gn}(s) \neq \mathrm{gn}(s')$. By Theorem A, $\bar{G}$ contains a hamiltonian path P, say P: $v_1, v_2, \ldots, v_n$. Let c be the greedy radio labeling of s: $v_1, v_2, \ldots, v_n$. Then $c(v_i) = i$ for all $1 \leqslant i \leqslant n$ and so $\mathrm{gn}(s) = n$.

On the other hand, $\bar{G}$ contains paths that cannot be extended to a hamiltonian path in $\bar{G}$. Among all such paths, let P': $u_1, u_2, \ldots, u_\ell$ $(\ell < n)$ be a path of greatest length in $\bar{G}$. Let $u_{\ell+1}, u_{\ell+2}, \ldots, u_n$ be the remaining vertices of G. Then u_ℓ is not adjacent to u_j in $\bar{G}$ for all j with $\ell + 1 \leqslant j \leqslant n$. Let s': $u_1, u_2, \ldots, u_n$ and let c' be the greedy radio labeling of s'. Then $c'(u_i) = i$ for all $1 \leqslant i \leqslant \ell$. For each j with $\ell + 1 \leqslant j \leqslant n$, since u_ℓ is not adjacent to u_j in $\bar{G}$ and $c'(u_\ell) = \ell$, it follows that $u_\ell u_j$ is an edge in G and so $c'(u_j) \geqslant \ell + 2$. Moreover, all labels $c'(u_j)$ $(\ell + 1 \leqslant j \leqslant n)$ are distinct. Hence $\mathrm{gn}(s') \geqslant n + 1$ and so $\mathrm{gn}(s) \neq \mathrm{gn}(s')$, producing the desired result. $\square$

The converse of Theorem 3.1 is false, however. For example, consider the 5-cycle C_5: $v_1, v_2, v_3, v_4, v_5, v_1$. Then $\bar{C}_5 = C_5$. Let s: v_1, v_2, v_3, v_4, v_5 and s': v_1, v_4, v_3, v_2, v_5; and let c and c' be the greedy labelings of s and s', respectively. Since $c(v_1) = 1$, $c(v_2) = 3$, $c(v_3) = 5$, $c(v_4) = 2$, and $c(v_5) = 4$, it follows that $\mathrm{gn}(s) = 5$. On the other hand, $c'(v_1) = 1$, $c'(v_4) = 2$, $c'(v_3) = 4$, $c'(v_2) = 6$, and $c'(v_5) = 5$, implying that $\mathrm{gn}(s') = 6$. The greedy labelings c and c' are shown in Fig. 3. Thus $\mathrm{GN}(C_5) > \mathrm{gn}(C_5)$. However, every path in $\bar{C}_5 = C_5$ can be extended to a hamiltonian path in C_5.

We now turn our attention to complete multipartite graphs (that are not complete). The following result appeared in [1].

Theorem C. *For a complete t-partite graph $K_{n_1, n_2, \ldots, n_t}$, where $t \geqslant 2$,*

$$\mathrm{rn}(K_{n_1, n_2, \ldots, n_t}) = \left(\sum_{i=1}^{t} n_i \right) + (t - 1).$$

By Theorems 2.1 and C, we have

$$\mathrm{gn}(K_{n_1, n_2, \ldots, n_t}) = \left(\sum_{i=1}^{t} n_i \right) + (t - 1).$$

Next, we determine the Grundy radio number of all complete multipartite graphs. A *linear forest* is a graph all of whose components are paths.

Theorem 3.2. *For any complete multipartite graph G of order n that is not a complete graph,*

$$\mathrm{GN}(G) = n + f,$$

where f is the largest size of a linear spanning forest of G.

Proof. Let $G = K_{n_1,n_2,\dots,n_t}$, where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$ with $t \geqslant 2$ and $n = \sum_{i=1}^{t} n_i$. Also, let $V_1, V_2, \dots, V_t$ be the partite sets of G, where $|V_i| = n_i$ for $1 \leqslant i \leqslant t$. Let $s : w_1, w_2, \dots, w_n$ be an ordering of $V(G)$ and let c be the greedy labeling of s. By a *block* of s, we mean a set $B = \{w_\alpha, w_{\alpha+1}, \dots, w_\beta\}$, where $1 \leqslant \alpha \leqslant \beta$, such that $B \subseteq V_j$ for some j $(1 \leqslant j \leqslant t)$, $w_{\alpha-1} \notin V_j$ if $\alpha > 1$, and $w_{\beta+1} \notin V_j$ if $\beta < n$. Let $B_1, B_2, \dots, B_k$ be the distinct blocks of s, where $t \leqslant k \leqslant f+1$, such that if $w_a \in B_p$ and $w_b \in B_q$, where $p < q$, then $a < b$. There are $k-1$ distinct pairs $\{w_i, w_{i+1}\}$ such that w_i and w_{i+1} belong to different blocks of s and $(n-1)-(k-1) = n-k$ distinct pairs $\{w_i, w_{i+1}\}$ such that w_i and w_{i+1} belong to the same block of s.

By definition, $c(w_1) = 1$. We claim that for each i with $1 \leqslant i \leqslant n-1$, (a) $c(w_{i+1}) = c(w_i) + 1$ if w_i and w_{i+1} belong to the same block of s and (b) $c(w_{i+1}) = c(w_i) + 2$ if w_i and w_{i+1} belong to different blocks of s. Assume, to the contrary, that c does not satisfy both (a) and (b). Then there exists a smallest integer j such that either (a) or (b) fails. We consider these two cases.

Case 1: (a) *fails*. Thus w_j and w_{j+1} belong to the same block of s and so belong to the same partite set, say V_p $(1 \leqslant p \leqslant t)$, and $c(w_{j+1}) \neq c(w_j) + 1$. Since $c(w_1), c(w_2), \dots, c(w_j)$ is an increasing sequence and c is a greedy radio labeling, it is not the case that $c(w_{j+1}) > c(w_j) + 1$. This implies that $c(w_{j+1}) < c(w_j)$. Let $c(w_{j+1}) = \ell$. So $2 \leqslant \ell < c(w_j)$. Let r be the greatest integer such that $r < j$ for which $c(w_r) < \ell$. Thus $c(w_{r+1}) \geqslant \ell$. Since $w_{r+1} \neq w_{j+1}$, it follows that $c(w_{r+1}) > \ell$ and so $c(w_{r+1}) = c(w_r) + 1$ or $c(w_{r+1}) = c(w_r) + 2$, according to whether w_r and w_{r+1} belong to the same partite set or distinct partite sets, respectively. However, since $c(w_r) < \ell$ and $c(w_{r+1}) > \ell$, it follows that $c(w_r) = \ell - 1$ and $c(w_{r+1}) = \ell + 1$. This implies that w_r and w_{r+1} belong to different partite sets of G. Since either w_r or w_{r+1} does not belong to V_p, either $|c(w_r) - c(w_{j+1})| \geqslant 2$ or $|c(w_{r+1}) - c(w_{j+1})| \geqslant 2$, a contradiction.

Case 2: (b) *fails*. Thus, w_j and w_{j+1} belong to distinct blocks and so to distinct partite sets of G, say $w_j \in V_p$ and $w_{j+1} \in V_q$, where $1 \leqslant p \neq q \leqslant t$, and $c(w_{j+1}) \neq c(w_j) + 2$. Since $c(w_1), c(w_2), \dots, c(w_j)$ is an increasing sequence and c is a greedy radio labeling, it is not the case that $c(w_{j+1}) > c(w_j) + 2$. This implies that $c(w_{j+1}) \leqslant c(w_j) + 1$. Since $w_j w_{j+1} \in E(G)$, it follows that $|c(w_{j+1}) - c(w_j)| \geqslant 2$, which yields $c(w_{j+1}) \leqslant c(w_j) - 2$. Let $c(w_{j+1}) = \ell$. So $2 \leqslant \ell \leqslant c(w_j) - 2$. Let r be the greatest integer with $r < j$ for which $c(w_r) < \ell$. Thus $c(w_{r+1}) \geqslant \ell$. Since $w_{r+1} \neq w_{j+1}$, it follows that $c(w_{r+1}) > \ell$. As in Case 1, this implies that $c(w_r) = \ell - 1$ and $c(w_{r+1}) = \ell + 1$. Since c is a greedy radio labeling, w_r and w_{r+1} belong to distinct partite sets of G, at least one of which is not V_p. Thus, either $|c(w_r) - c(w_{j+1})| \geqslant 2$ or $|c(w_{r+1}) - c(w_{j+1})| \geqslant 2$, a contradiction.

Since c satisfies (a) and (b), it follows that

$$\mathrm{gn}(s) = c(w_n) = 1 + 2(k-1) + (n-k) = n + k - 1.$$

Since $k \leqslant f+1$, it follows that $\mathrm{gn}(s)$ is maximized when $k = f+1$ and so $\mathrm{GN}(G) = n + (f+1) - 1 = n + f$. $\square$

It is straightforward to show the following result.

Proposition 3.3. *Let $G = K_{n_1,n_2,\ldots,n_t}$, where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, $t \geqslant 2$, and $n_t \geqslant 2$. Let $n = \sum_{i=1}^{t} n_i$ and $n' = \sum_{i=1}^{t-1} n_i$. Then the maximum size of a spanning linear forest in G is $n-1$ if $n_t \leqslant n'+1$ and $2n'$ if $n_t \geqslant n'+2$.*

Theorem 3.2 and Proposition 3.3 now yield the following corollaries.

Corollary 3.4. *Let $G = K_{n_1,n_2,\ldots,n_t}$, where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, $t \geqslant 2$, and $n_t \geqslant 2$. Furthermore, let $n = \sum_{i=1}^{t} n_i$ and $n' = \sum_{i=1}^{t-1} n_i$. Then*

$$\mathrm{GN}(G) = \begin{cases} 2n-1 & \text{if } n_t \leqslant n'+1, \\ n+2n' & \text{if } n_t \geqslant n'+2. \end{cases}$$

Corollary 3.5. *For a complete bipartite graph $K_{a,b}$ of order at least 3,*

$$\mathrm{GN}(K_{a,b}) = \begin{cases} 3a+b & \text{if } 1 \leqslant a < b, \\ 4a-1 & \text{if } 2 \leqslant a = b. \end{cases}$$

It is not difficult to see that $\mathrm{gn}(K_n) = \mathrm{GN}(K_n) = \chi(K_n) = n$ for every integer $n \geqslant 2$. By Theorem C, $\mathrm{gn}(K_n - e) = 2n-2$ for every edge e of K_n, while $\mathrm{GN}(K_n - e) = 2n-1$. By Corollary 3.5, $\mathrm{gn}(K_{1,n-1}) = n+1$ and $\mathrm{GN}(K_{1,n-1}) = n+2$ for $n \geqslant 3$. So there are two classes of graphs G for which $\mathrm{GN}(G) = \mathrm{gn}(G) + 1$. There is reason to believe that if G is any connected graph that is not complete, then $\mathrm{GN}(G) > \mathrm{gn}(G)$.

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